Effect of thermal radiation on the propagation of plane acoustic waves

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A study is made of the propagation of acoustic waves in a semi-infinite expanse of radiating gas on one side of an infinite, plane, radiating wall. A solution is found, in particular, for the case of sinusoidal oscillations in both position and temperature of the wall. The solution is based on a single linear integro-differential equation that plays the same role here as does the classical wave equation in equilibrium acoustic theory. The solution is applicable throughout the range from a completely transparent to a completely opaque gas and from very low to very high temperatures. The solution appears, in general, as the sum of two types of travelling waves: (1) an essentially classical sound-wave, but with a slightly altered speed and a small amount of damping and (2) a radiation-induced wave whose speed and damping may be either large or small, depending on the temperature and absorptivity of the gas. Since the waves are coupled, both types will usually be present together, even in the special cases of pure motion or pure temperature variation of the wall.

Introduction

This work arose out of a desire to learn something about the non-equilibrium interaction between thermal radiation and fluid flow in a case in which the mathematics would not be too difficult. As in earlier investigations of chemical and vibrational non-equilibrium (see, for example, Moore 1958, Broer 1958, Clarke 1958, and Vincenti 1959), the best possibility appeared to lie in the study of acoustic waves in one dimension. It transpires, in fact, that interesting results can be obtained once more with nothing more sophisticated than complex algebra, though the algebraic manipulations are at times rather tedious.

To fix the problem, we consider in particular a semi-infinite expanse of radiating gas on one side of an infinite, plane, radiating wall and inquire as to what disturbances are caused in the gas by small sinusoidal oscillations in both the position and temperature of the wall. In so far as its one-dimensional nature is concerned, this problem has much in common with the classical astrophysical problem of the plane-parallel stellar atmosphere (see, for example, Chandrasekhar 1950 or Kourganoff 1952). In astrophysical problems, however, the motion of the gas may be neglected, and there is no occasion to consider the influence of solid walls. Recently, several authors with aerodynamic interests (Goulard 1959*b*; Goulard & Goulard 1959; Tellep & Edwards 1960) have extended the treatment of radiation in the plane-parallel case to include the effects of both fluid motion and solid boundaries, but primarily with the study of boundary-layer heat-transfer in mind.

So far as the authors are aware, the existing original literature on radiative effects in acoustic propagation is limited to work by Stokes (1851) and by Prokofyev (1957, 1960). Stokes's paper, written over 100 years ago, was part of a vigorous controversy then going on in the pages of the *Philosophical Magazine* with regard to the validity of Laplace's adiabatic theory of sound as against Newton's isothermal theory. Stokes, on the basis of approximations appropriate to highly transparent, low-temperature air, was able to show that radiation could not have an appreciable effect under ordinary atmospheric conditions, and hence that Laplace's adiabatic hypothesis could not be called into question on that account. (For a brief outline of Stokes's work, see also Rayleigh 1945, vol. II, p. 24.) Prokofyev's work, which came to the authors' attention after the present study was nearly complete, considers the problem of thermally radiating acoustic waves in great generality, including the effects of viscosity and thermal conductivity as well as the smaller effects (for aerodynamic purposes, at least) of radiation scattering, radiation pressure, and the direct contribution of radiation to internal energy. Specific results are restricted to certain extreme values of the variables, however; and the important relationship between the waves and the boundary conditions is not taken into account.

To simplify the problem and to isolate the influence of radiation, we assume here that non-equilibrium effects from other processes, such as molecular transport, dissociation, vibration, etc., are negligible. Radiation scattering, radiation pressure, and the contribution of radiation to internal energy are also neglected. For simplicity, the gas is assumed to be perfect. The radiative effects are taken into account on the basis of the usual quasi-equilibrium hypothesis and the assumption of a 'grey gas', that is, that the absorption coefficient is independent of wavelength. The last assumption is, however, not essential.

On the foregoing basis and with the approximations usual in acoustic theory, the fundamental equations combine to give a single linear integro-differential equation for a disturbance-velocity potential. This equation, which appears as equation (38), includes the emission and reflexion effects of the wall and is applicable over the complete range of the radiation parameters. It plays the same basic role here as does the one-dimensional wave equation in classical equilibrium acoustic theory. With the aid of a suitable approximation to the attenuation function appearing in the radiation terms, the equation is solved for the case of a black wall undergoing sinusoidal disturbances. The solution is found to appear, in general, as the sum of two waves: (1) a slightly modified version of the classical sound wave, and (2) a radiation-induced wave that has no counterpart in classical acoustic theory.

Basic equations

We begin by writing down the equations for three-dimensional time-dependent flow neglecting molecular-transport terms but including the effect of radiative transfer of heat (see, for example, Hirschfelder, Curtiss & Bird 1954 or Lighthill 1960). In suffix notation, with the usual convention that repeated dummy subscripts denote summation, the equations for conservation of mass, momentum, and energy are respectively

$$\frac{D\rho}{Dt}+\rho\frac{\partial u_j}{\partial x_j}=0, \tag{1}$$

$$\rho\frac{Du_i}{Dt}+\frac{\partial p}{\partial x_i}=0 \quad (i=1,2,3) \tag{2}$$

and

$$\rho\frac{Dh}{Dt}-\frac{Dp}{Dt}=Q. \tag{3}$$

Here ρ is the mass density, u_i $(i = 1,2,3)$ the velocity component in the x_i direction, p the pressure, h the enthalpy per unit mass, and Q the net rate of heat input to the gas per unit volume as the result of radiation. The derivative following a fluid element is given as usual by $D(\ )/Dt \equiv \partial(\ )/\partial t+u_j\partial(\ )/\partial x_j$.

In line with the neglect of all non-equilibrium effects other than radiation, the enthalpy is related to the other state variables by an equilibrium thermodynamic relation of the form $h = h(p,\rho)$. In particular, for the perfect gas assumed here we have

$$h=\frac{\gamma}{\gamma-1}\frac{p}{\rho}, \tag{4}$$

where γ is the (constant) ratio of specific heats. The temperature T, which will also enter later in the evaluation of Q, is given correspondingly by

$$T=p/R\rho, \tag{5}$$

where R is the gas constant per unit mass.

In accord with the assumptions of the introduction, the foregoing system of equations neglects the small direct contribution of the radiation to the fluid pressure and to the internal energy of the gas. This is formally equivalent to assuming that the speed of propagation of radiation is infinite. With this assumption, the only effect of radiation is the appearance of the rate of heat transfer Q in the energy equation (3). (For a system of flow equations that takes account of radiation pressure and internal radiant energy, see Prokofyev 1957.)

To evaluate Q, recourse must be had to the theory of radiant heat transfer. (Among the several texts excellent for this purpose, primarily by astrophysicists, are those of Chandrasekhar 1950, Kourganoff 1952, and Unsöld 1955; a useful review for aerodynamicists is given in the paper by Lighthill 1960.) If Q_ν is the net rate of heat input per unit frequency per unit volume by radiation in the frequency range from ν to $\nu+d\nu$, then the value of Q at a point can be written

$$Q=\int_0^\infty Q_\nu\,d\nu=\int_0^\infty (A_\nu-E_\nu)\,d\nu, \tag{6}$$

where A_ν and E_ν are respectively the rate of absorption and emission of radiation in the aforementioned range.

To evaluate A_ν at a point it is necessary to consider the radiation intensity, which varies not only from point to point, but also with direction through any given point. If a particular direction through a given point is identified by the symbol Ω, then the intensity of radiation through that point in the direction Ω and in the frequency range ν to $\nu+d\nu$ can be specified by the value of the specific intensity $I_\nu(\Omega)$, which is by definition the rate of energy transfer per unit frequency, per unit solid angle in the given direction, and per unit area normal to that direction. Now let the absorption coefficient α_ν denote the proportion of the energy of a radiant beam in the frequency range ν to $\nu+d\nu$ that is absorbed by the gas per unit distance. The rate of absorption A_ν per unit volume at a point can then be shown to be given by

$$A_\nu = \alpha_\nu \int_0^{4\pi} I_\nu(\Omega)\, d\Omega, \tag{7}$$

where the integration extends over all of the elements of solid angle $d\Omega$ about the point in question. The absorption coefficient α_ν is a function of the local state of the gas; in the usual quasi-equilibrium theory, it is taken to have the value appropriate to complete equilibrium at the local thermodynamic state.

In accordance with the assumption of quasi-equilibrium, the rate of emission E_ν per unit volume is taken as

$$E_\nu = 4\pi\alpha_\nu B_\nu(T). \tag{8a}$$

Here $B_\nu(T)$ is the equilibrium intensity given by the Planck function

$$B_\nu(T) \equiv \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}, \tag{8b}$$

where T is the local temperature, c the speed of light, and h and k the Planck and Boltzmann constants, respectively.

To find the specific intensity $I_\nu(\Omega)$ required in equation (7), it is necessary to solve the so-called 'equation of transfer', which is the basic differential equation governing the intensity of radiation in a given direction in a gas in radiative non-equilibrium (see, for example, Chandrasekhar 1950). If scattering by the gas is neglected, the formal solution appropriate to the earlier assumption of an infinite speed of light can be written

$$I_\nu(\Omega) = I_\nu(r_S, \Omega)\exp\{-\tau_\nu(r_S, \Omega)\} + \int_0^{r_S} \alpha_\nu(r, \Omega)\, B_\nu(r, \Omega)\exp\{-\tau_\nu(r, \Omega)\}\, dr. \tag{9a}$$

This relation gives the specific intensity at a point in the field and in a given direction Ω as the sum of two parts: (1) the specific intensity from a solid surface a radial distance r_S away from the point in the direction opposite to Ω, attenuated by an exponential factor $\exp\{-\tau_\nu(r_S, \Omega)\}$ to account for absorption by the intervening gas, and (2) the total effect of the emission from elements of gas a variable distance r in the direction opposite to Ω, each elementary contribution attenuated by the appropriate factor $\exp\{-\tau_\nu(r, \Omega)\}$ and the whole summed over all elements from the point to the solid surface. The quantity $\tau_\nu(r, \Omega)$, called the 'optical thickness' of the gas over the radial distance r from the point, is given by

$$\tau_\nu(r, \Omega) \equiv \int_0^r \alpha_\nu(\hat{r}, \Omega)\, d\hat{r}. \tag{9b}$$

The contribution Q_ν to the integral of equation (6) can now be found by substitution from equations (7), (8a), and (9a). The result, after changing the radial variable of integration from r to τ_ν, can be written

$$Q_\nu = \alpha_\nu \left\{ \int_0^{4\pi} d\Omega \left[I_\nu(\tau_{\nu S})\, e^{-\tau_{\nu S}} + \int_0^{\tau_{\nu S}} B_\nu\, e^{-\tau_\nu}\, d\tau_\nu \right] - 4\pi B_\nu \right\}. \tag{10}$$

When supplemented by this equation and equations (6), (8b) and (9b), equations (1) to (5) constitute a set of seven equations for the seven unknowns ρ, u_i, p, h, and T (assuming, of course, that $I_\nu(\tau_{\nu S})$ and α_ν are known).

In working with the foregoing equations it will also be useful to have an expression for the rate of heat transfer through an infinitesimal element of area with given orientation within the fluid. If q_ν is the rate of heat flow through the element per unit area per unit frequency and in the direction of a chosen normal to the element, and if χ is the angle between that normal and the direction Ω, then we can write

$$q_\nu = \int_0^{4\pi} I_\nu(\Omega) \cos\chi\, d\Omega. \tag{11}$$

After substitution from equation (9a) this can be written

$$q_\nu = \int_0^{4\pi} \cos\chi\, d\Omega \left[I_\nu(\tau_{\nu S})\, e^{-\tau_{\nu S}} + \int_0^{\tau_{\nu S}} B_\nu\, e^{-\tau_\nu}\, d\tau_\nu \right]. \tag{12}$$

In applying this equation, care must be taken to reckon the angle χ correctly, i.e. to the proper direction Ω, which is *opposite* to the direction along which the integration with respect to τ_ν is made.

One-dimensional unsteady flow

The next step is to simplify the foregoing equations to the case in which conditions are a function of only one Cartesian space variable, x say. The simplification of the differential terms in equations (1), (2) and (3) is obvious and need not be discussed. The transfer term Q, however, requires some attention, particularly as regards the effect of the wall.

To fix our ideas, let us consider a semi-infinite expanse of gas on one side of a solid wall perpendicular to the x-axis and located at $x = x_w$ (see figure 1). Later x_w will be regarded as a function of time; but, since the situation is assumed to be quasi-steady as regards radiation (i.e. assuming an infinite speed of light), this is of no importance in the treatment of Q itself. Let the direction through the point at x be defined by the angles θ and λ as shown—that is, r, θ, λ are spherical polar co-ordinates centred on the point. The element of solid angle can then be written as $d\Omega = \sin\theta\, d\theta\, d\lambda$ or, if $\mu = \cos\theta$, as $d\Omega = -d\mu\, d\lambda$. The co-ordinate $\tilde{x}$ of any other point in the field is given by $\tilde{x} = x + r\cos\theta = x + r\mu$, so that for a fixed value of x we have $dr = d\tilde{x}/\mu$. With this transformation, equation (9*b*) for τ_ν can be written

$$\tau_\nu = \frac{1}{\mu}\int_x^{\tilde{x}} \alpha_\nu \, d\hat{x},$$

or, since α_ν is a function only of x in the present instance,

$$\tau_\nu(x, \tilde{x}, \mu) = \{\eta_\nu(\tilde{x}) - \eta_\nu(x)\}/\mu = (\tilde{\eta}_\nu - \eta_\nu)/\mu,$$

where
$$\eta_\nu = \eta_\nu(x) \equiv \int_{x_w}^{x} \alpha_\nu(\hat{x})\, d\hat{x}. \tag{13}$$

The quantity $\eta_\nu(x)$ is the optical thickness, parallel to the x-axis, of the gas between the wall and the station x.

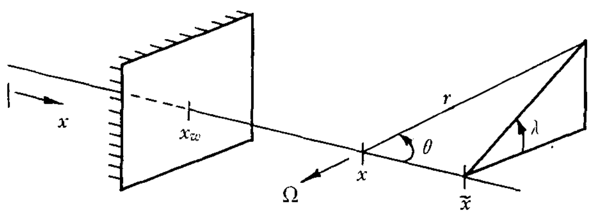

FIGURE 1. The co-ordinate system.

In terms of the foregoing variables, the integral over $d\Omega$ in equation (10) which we denote by F_ν, can now be written

$$F_\nu = -\int_0^{2\pi} d\lambda \int_{+1}^{-1} d\mu \left[I_\nu(\eta_{\nu S}, \mu) \exp\{-(\eta_{\nu S} - \eta_\nu)/\mu\} + \int_{\eta_\nu}^{\eta_{\nu S}} \mu^{-1} B_\nu \exp\{-(\tilde{\eta}_\nu - \eta_\nu)/\mu\}\, d\tilde{\eta}_\nu \right],$$

where $\eta_{\nu S} = \eta_\nu(x_S)$ specifies the location of any solid surface that is perpendicular to the x-axis. Since all of the quantities in the integrand are independent of λ in the present case, the integration with respect to this variable can be carried out at once and provides a factor 2π. In the integration with respect to μ, a distinction is necessary between the ranges $+1$ to 0 and 0 to -1. In the former range no solid surface is encountered and hence $\eta_{\nu S} = \infty$. In the latter range, the wall is present at $x = x_w$, and we have $\eta_{\nu S} = \eta_\nu(x_w) = 0$. Dividing the integration with respect to μ into these two ranges and interchanging the order of integration in the integral involving B_ν, we arrive in the end at the following result (where for the sake of uniformity the integration over μ from 0 to -1 has been formally replaced by an integration over $-\mu$ from 0 to $+1$):

$$F_\nu = 2\pi\left\{\int_0^1 I_\nu(0, -\mu)\, e^{-\eta_\nu/\mu}\, d\mu + \int_0^{\eta_\nu} B_\nu E_1(\eta_\nu - \tilde{\eta}_\nu)\, d\tilde{\eta}_\nu + \int_{\eta_\nu}^{\infty} B_\nu E_1(\tilde{\eta}_\nu - \eta_\nu)\, d\tilde{\eta}_\nu\right\}. \tag{14a}$$

The function $E_1(z)$ that appears here is a particular case of the integro-exponential function of order n, defined by

$$E_n(z) \equiv \int_0^1 \mu^{n-2} e^{-z/\mu}\, d\mu. \tag{14b}$$

The properties of this function are discussed and numerical values tabulated in the text by Kourganoff (1952), pp. 253 and 266. Note that in the present one-dimensional case, B_ν is a function only of x or alternatively only of η_ν, which is the reason that it can be taken outside the integration with respect to μ.

It remains to evaluate $I_\nu(0, -\mu)$, the specific intensity at the wall in equation (14*a*). To this end, it is written as the sum of three parts, due respectively to emission, specular reflexion, and diffuse reflexion, or

$$I_\nu(0, -\mu) = I_{\nu_e}(0, -\mu) + I_{\nu_s}(0, -\mu) + I_{\nu_d}(0, -\mu). \tag{15}$$

These will be considered in order.

(*a*) According to Lambert's law, the specific intensity of emitted radiation is independent of direction, i.e. of $-\mu$. In line with the quasi-equilibrium assumption it can be written

$$I_{\nu_e}(0, -\mu) \equiv I_{\nu_e}(0) = \epsilon_\nu B_\nu(T_w), \tag{16}$$

where ϵ_ν is the emissivity of the wall for radiation of frequency ν, and T_w is the wall temperature.

(*b*) The specific intensity $I_{\nu_s}(0, -\mu)$ of specularly reflected radiation does depend on direction. It can be found by multiplying the corresponding incoming intensity $I_\nu(0, \mu)$ by the fraction r_{ν_s} of incoming radiation of frequency ν that is specularly reflected. We thus obtain, with the aid of equation (9*a*) and the subsequent changes of variable,

$$I_{\nu_s}(0, -\mu) = r_{\nu_s} I_\nu(0, \mu) = r_{\nu_s} \int_0^\infty \mu^{-1} B_\nu e^{-\tilde{\eta}_\nu/\mu} \, d\tilde{\eta}_\nu. \tag{17}$$

Strictly speaking, the specular reflectivity r_{ν_s} may itself be a function of direction, but we shall ignore this complication and assume it to be independent of μ.

(*c*) To find the intensity of diffuse reflexion, which is by definition independent of μ, we first specialize the general equation (12) to find the heat flux at a point on the wall. If the inward normal to the wall is taken as the pertinent one, then for the present one-dimensional case we have $\cos\chi = \cos\theta = \mu$. Transforming equation (12) into the new variables and handling the integrations in the same manner as for the integral of equation (10), we obtain finally for a point on the wall ($\eta_\nu = 0$)

$$q_{\nu w} = 2\pi \left\{ -\int_0^1 I_\nu(0, -\mu) \, \mu \, d\mu + \int_0^\infty B_\nu E_2(\tilde{\eta}_\nu) \, d\tilde{\eta}_\nu \right\}. \tag{18}$$

In this equation for the net heat flux to the wall, the second integral is the flux into the wall from the gas, while the first integral is the flux out of the wall to the gas. If r_{ν_d} is the fraction of the incoming radiation in a given direction μ that is diffusely reflected, and if it is assumed that this fraction, like r_{ν_s}, is independent of μ, then the portion of the incoming flux that is diffusely reflected is given by the second integral multiplied by r_{ν_d}. The portion of the outgoing flux that arises from the diffuse reflexion is also represented by the first integral if we put $I_{\nu_d}(0, -\mu)$ in place of $I_\nu(0, -\mu)$. Equating these two expressions and utilizing the fact that $I_{\nu_d}(0, -\mu)$ is actually independent of μ, we thus obtain

$$I_{\nu_d}(0, -\mu) = I_{\nu_d}(0) = 2r_{\nu_d} \int_0^\infty B_\nu E_2(\tilde{\eta}_\nu) \, d\tilde{\eta}_\nu. \tag{19}$$

The final result for Q_ν is now found by collecting the expressions (16), (17), and (19), substituting the result into equation (14a), and then using the expression thus obtained for F_ν to replace the integral term in equation (10). We thus obtain finally for the present one-dimensional situation with a single wall

$$Q_\nu = 2\pi\alpha_\nu\Bigg\{\left[\epsilon_\nu B_\nu(T_w) + 2r_{\nu_d}\int_0^\infty B_\nu E_2(\tilde{\eta}_\nu)\,d\tilde{\eta}_\nu\right]E_2(\eta_\nu)$$
$$+ r_{\nu_s}\int_0^\infty B_\nu E_1(\tilde{\eta}_\nu+\eta_\nu)\,d\tilde{\eta}_\nu + \int_0^{\eta_\nu} B_\nu E_1(\eta_\nu-\tilde{\eta}_\nu)\,d\tilde{\eta}_\nu$$
$$+ \int_{\eta_\nu}^\infty B_\nu E_1(\tilde{\eta}_\nu-\eta_\nu)\,d\tilde{\eta}_\nu - 2B_\nu\Bigg\}. \quad (20)$$

Of the various terms on the right, those containing ϵ_ν, r_{ν_d}, and r_{ν_s} represent the radiative heat input to an element of gas at a point η_ν due respectively to emission, diffuse reflexion, and specular reflexion from the wall; the next two terms represent the heat input from the other elements of gas, the first from the elements to the left of η_ν and the second from the elements to the right; the last term represents the heat loss from the element by emitted radiation. Equation (20) agrees with relations given earlier by Goulard (1959b), who, however, did not include the effect of specular reflexion (see also Tellep & Edwards 1960). The formalism here, however, is considerably different.

In the special case of complete equilibrium, $B_\nu(T)$ is everywhere the same and Q_ν must vanish. Application of these conditions in equation (20) and use of the properties of the functions $E_n(z)$ lead to the known result that at equilibrium (and in the present case in which r_{ν_d} and r_{ν_s} are independent of the direction of the incident radiation)

$$\epsilon_\nu = 1 - r_{\nu_d} - r_{\nu_s}. \quad (21)$$

Consistent with the basic quasi-equilibrium assumption, this relation is also taken to be true in the non-equilibrium situation, with the same numerical values of the various quantities as apply at equilibrium.

Grey-gas approximation

To find the total value of Q for use in the flow equations, it is necessary to integrate equation (20) over all values of ν from 0 to ∞ (cf. equation (6)). This leads to complicated double integrals that make further progress difficult. In astrophysics it has been found fruitful to side-step this difficulty by introducing the so-called 'grey-gas approximation'. The same procedure will be followed here. A less restrictive method that leads to qualitatively the same results is outlined in the concluding section.

The grey-gas approximation consists in replacing the absorption coefficient α_ν by a constant value α independent of ν. With this approximation, η_ν as given by equation (13) is also independent of ν, i.e.

$$\eta(x) = \int_{x_w}^x \alpha(\hat{x})\,d\hat{x}. \quad (22)$$

If ϵ_ν, r_{ν_d}, and r_{ν_s} are also assumed independent of ν, equation (20) can then be readily integrated with the aid of Stefan's law,

$$\int_0^\infty B_\nu d\nu = \frac{\sigma}{\pi} T^4, \tag{23}$$

where σ is the Stefan–Boltzmann constant. We thus obtain

$$Q = \int_0^\infty Q_\nu d\nu = 2\sigma\alpha\left\{\left[\epsilon T_w^4 + 2r_d \int_0^\infty T^4 E_2(\tilde{\eta})\, d\tilde{\eta}\right] E_2(\eta) + r_s \int_0^\infty T^4 E_1(\tilde{\eta}+\eta)\, d\tilde{\eta} + \int_0^\eta T^4 E_1(\eta-\tilde{\eta})\, d\tilde{\eta} + \int_\eta^\infty T^4 E_1(\tilde{\eta}-\eta)\, d\tilde{\eta} - 2T^4\right\}. \tag{24}$$

The foregoing relation can be expressed in a useful alternative form by integrating each integral once by parts. In doing this, use is made of the general differentiation formula $dE_n(z)/dz = -E_{n-1}(z)$ and of the special values $E_2(0) = 1$, $E_3(0) = \frac{1}{2}$, and $E_2(\infty) = E_3(\infty) = 0$. With the aid of the relation (21), we thus obtain finally

$$Q = 2\sigma\alpha\left\{\left[\epsilon(T_w^4 - T_{x=x_w}^4) + 8r_d \int_{\tilde{\eta}=0}^{\tilde{\eta}=\infty} T^3 E_3(\tilde{\eta})\, dT\right] E_2(\eta) + 4r_s \int_{\tilde{\eta}=0}^{\tilde{\eta}=\infty} T^3 E_2(\tilde{\eta}+\eta)\, dT - 4\int_{\tilde{\eta}=0}^{\tilde{\eta}=\eta} T^3 E_2(\eta-\tilde{\eta})\, dT + 4\int_{\tilde{\eta}=\eta}^{\tilde{\eta}=\infty} T^3 E_2(\tilde{\eta}-\eta)\, dT\right\}. \tag{25}$$

Here $T_{x=x_w}$ is the temperature of the gas immediately adjacent to the wall, which need not be the same as that of the wall itself in the assumed absence of molecular transport processes. Equation (25) shows clearly that Q vanishes when T is everywhere the same.

Acoustic equations

We now proceed to the linearized approximation to the foregoing equations. To this end, we consider disturbances on a uniform gas at rest and write $u = u'$, $p = p_0 + p'$, $T = T_0 + T'$, etc., where the subscript 0 denotes the uniform conditions in the undisturbed gas and the primes denote the disturbances therefrom, which are to be regarded as small. For the undisturbed field to be uniform, it must, of course, be in radiative equilibrium ($Q_0 = 0$); in particular, the undisturbed wall temperature T_{w0} must be equal to the uniform gas temperature T_0.

The linearization of the differential terms in the flow equations proceeds straightforwardly in the usual fashion. In the assumed one-dimensional situation, we obtain, as the linearized approximation to the conservation equations (1), (2) and (3),

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial u'}{\partial x} = 0, \tag{26}$$

$$\rho_0 \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0, \tag{27}$$

and

$$\rho_0 \frac{\partial h'}{\partial t} - \frac{\partial p'}{\partial t} = Q'. \tag{28}$$

The linearized forms of the state relations (4) and (5) are

$$h' = \frac{\gamma}{\gamma-1}\left(\frac{1}{\rho_0}p' - \frac{p_0}{\rho_0^2}\rho'\right) \tag{29}$$

and

$$T' = \frac{1}{R}\left(\frac{1}{\rho_0}p' - \frac{p_0}{\rho_0^2}\rho'\right). \tag{30}$$

The linearization of the radiative-transfer term Q requires somewhat more attention. To begin, we note that the absorption coefficient, being a function of the local thermodynamic state, can also be written in the form $\alpha = \alpha_0 + \alpha'$. Equation (22) for the optical thickness η then becomes

$$\eta(x) = \int_{x_w(t)}^{x} (\alpha_0 + \alpha')\, d\hat{x} = \alpha_0[x - x_w(t)] + \int_{x_w(t)}^{x} \alpha'\, d\hat{x},$$

where explicit recognition has now been taken of the fact that the displacement x_w of the wall from the origin is a function of time. If this displacement is assumed to be small, of the same order as the primed quantities, then η can also be written in the form

$$\eta = \alpha_0 x + \eta', \tag{31}$$

where

$$\eta' = -\alpha_0 x_w(t) + \int_{x_w(t)}^{x} \alpha'\, d\hat{x}.$$

The linearization is then most readily carried out on the basis of equation (25). Taking the integral from $\tilde{\eta} = 0$ to η as a typical term, we first expand $E_2(\eta - \tilde{\eta})$ as a Taylor series with the aid of equation (31), i.e. as

$$\begin{aligned} E_2(\eta - \tilde{\eta}) &= E_2[\alpha_0(x - \tilde{x}) + (\eta' - \tilde{\eta}')] \\ &= E_2[\alpha_0(x - \tilde{x})] + [dE_2(z)/dz]_{z=\alpha_0(x-\tilde{x})}(\eta' - \tilde{\eta}') + \dots \end{aligned} \tag{32}$$

Since $dT = dT'$, the integral can therefore be approximated to the first order in small quantities as

$$\int_{\tilde{\eta}=0}^{\tilde{\eta}=\eta} T^3 E_2(\eta - \tilde{\eta})\, dT = T_0^3 \int_{\tilde{\eta}=0}^{\tilde{\eta}=\eta} E_2[\alpha_0(x - \tilde{x})]\, dT' = T_0^3 \int_{x_w(t)}^{x} E_2[\alpha_0(x - \tilde{x})] \left(\frac{\partial T'}{\partial x}\right)_{\tilde{x}} d\tilde{x},$$

where dT' has been replaced on the right by $(\partial T'/\partial x)_{\tilde{x}}\, d\tilde{x}$, the subscript $\tilde{x}$ indicating that the derivative is to be evaluated as a function of the variable of integration $\tilde{x}$. Writing the final integral as an integral from 0 to x minus an integral from 0 to $x_w(t)$, and discarding the integral from 0 to $x_w(t)$ as being of second order in small quantities, we obtain finally

$$\int_{\tilde{\eta}=0}^{\tilde{\eta}=\eta} T^3 E_2(\eta - \tilde{\eta})\, dT = T_0^3 \int_0^x E_2[\alpha_0(x - \tilde{x})] \left(\frac{\partial T'}{\partial x}\right)_{\tilde{x}} d\tilde{x}.$$

By treating the other integrals in equation (25) in similar fashion and replacing $(T_w^4 - T_{x=x_w}^4)$ by the linear approximation $4T_0^3(T_w' - T_{x=0}')$, where conditions at the wall have been transferred to $x = 0$ as is usual in acoustic theory, we obtain the linearized form of the radiative-transfer term as follows

$$Q' = 8\sigma\alpha_0 T_0^3 \left\{ \left[\epsilon(T'_w - T'_{x=0}) + 2r_d \int_0^\infty E_3(\alpha_0 \tilde{x}) \left(\frac{\partial T'}{\partial x}\right)_{\tilde{x}} d\tilde{x} \right] E_2(\alpha_0 x) \right.$$
$$+ r_s \int_0^\infty E_2[\alpha_0(\tilde{x}+x)] \left(\frac{\partial T'}{\partial x}\right)_{\tilde{x}} d\tilde{x} - \int_0^x E_2[\alpha_0(x-\tilde{x})] \left(\frac{\partial T'}{\partial x}\right)_{\tilde{x}} d\tilde{x}$$
$$\left. + \int_x^\infty E_2[\alpha_0(\tilde{x}-x)] \left(\frac{\partial T'}{\partial x}\right)_{\tilde{x}} d\tilde{x} \right\}. \tag{33}$$

One difficulty in the foregoing development needs to be mentioned. This arises from the fact that the first and higher derivatives of $E_2(z)$ are singular at $z = 0$ (see Kourganoff 1952). This means that a series expansion such as that of equation (32) is not valid throughout the entire interval of integration—for example, at the limit $\tilde{x} = x$ in the integral considered above. Furthermore, because of the singular behaviour of the derivative of $E_2(z)$, it is to be expected from equation (25) that the derivatives of Q and hence of other physical quantities will be infinite at the wall ($\eta = 0$). It follows that difficulties must be encountered in justifying the transfer of boundary conditions from the wall to the origin, as is done both in classical acoustic theory and in the present paper. And finally, because of this entire situation, any attempt to extend the foregoing treatment beyond the linear approximation fails completely. A detailed re-examination of the whole problem shows that a valid systematic expansion procedure can be obtained by transforming from the geometrical co-ordinate x to the optical thickness η in the left-hand side of the conservation equations (1) to (3). The use of such a non-inertial and distorted co-ordinate system greatly complicates the fluid-mechanical terms in the equations but allows the expansion of the radiation term to be handled without difficulty. The details, however, are lengthy and tedious; and, fortunately for present purposes, the final result for the first approximation turns out to be the same as that obtained by the non-rigorous procedure given above. For this reason, a complete treatment of the matter will be deferred until a later time.

We now have in equations (26) to (30), when supplemented by equation (33) for Q', a set of five linear equations for the five unknowns ρ', u', p', h', and T'. They can be combined into a single integro-differential equation for a potential function as follows. First, the potential function ϕ is defined such that the momentum equation is satisfied, i.e. by

$$u' = \frac{\partial \phi}{\partial x}, \quad p' = -\rho_0 \frac{\partial \phi}{\partial t}. \tag{34}$$

Then, to find the governing equation for ϕ, we eliminate h' from the energy equation (28) by means of the state relation (29) and then replace $\partial \rho'/\partial t$ in the resulting equation by means of the continuity relation (26). This gives

$$\frac{\partial p'}{\partial t} + \gamma p_0 \frac{\partial u'}{\partial x} = (\gamma - 1)\, Q'(T')$$

or, after substitution from equations (34) and introduction of the speed of sound $a_0 = (\gamma p_0/\rho_0)^{\frac{1}{2}}$ for the assumed perfect gas,

$$\frac{\partial^2\phi}{\partial t^2} - a_0^2\frac{\partial^2\phi}{\partial x^2} = -\frac{(\gamma-1)}{\rho_0}Q'(T'). \tag{35}$$

To eliminate T' in the expression for Q' (cf. equation (33)), we differentiate the state relation (30) with respect to t and again replace $\partial\rho'/\partial t$ by means of the continuity relation (26). This gives

$$\frac{\partial T'}{\partial t} = \frac{1}{R\rho_0}\left(\frac{\partial p'}{\partial t} + p_0\frac{\partial u'}{\partial x}\right) = -\frac{1}{R}\left(\frac{\partial^2\phi}{\partial t^2} - \frac{a_0^2}{\gamma}\frac{\partial^2\phi}{\partial x^2}\right). \tag{36}$$

The final equation is now obtained by differentiating equation (35) with respect to t, after substituting for Q' from equation (33), and eliminating T' with the aid of equation (36). With the notation

$$W_s \equiv \frac{\partial^2\phi}{\partial t^2} - a_0^2\frac{\partial^2\phi}{\partial x^2}, \quad W_T \equiv \frac{\partial^2\phi}{\partial t^2} - \frac{a_0^2}{\gamma}\frac{\partial^2\phi}{\partial x^2}, \tag{37a,b}$$

the following equation is obtained as the integro-differential equation for the disturbance potential ϕ

$$\begin{aligned}\frac{\partial W_s}{\partial t} = 8\gamma\frac{(\gamma-1)\sigma T_0^3}{\gamma R\rho_0 a_0}\alpha_0 a_0\Bigg\{\Bigg[&-\epsilon\left(R\frac{dT'_w}{dt} + (W_T)_{x=0}\right) \\ &+ 2r_d\int_0^\infty E_3(\alpha_0\tilde{x})\left(\frac{\partial W_T}{\partial x}\right)_{\tilde{x}}d\tilde{x}\Bigg]E_2(\alpha_0 x) \\ &+ r_s\int_0^\infty E_2[\alpha_0(\tilde{x}+x)]\left(\frac{\partial W_T}{\partial x}\right)_{\tilde{x}}d\tilde{x} - \int_0^x E_2[\alpha_0(x-\tilde{x})]\left(\frac{\partial W_T}{\partial x}\right)_{\tilde{x}}d\tilde{x} \\ &+ \int_x^\infty E_2[\alpha_0(\tilde{x}-x)]\left(\frac{\partial W_T}{\partial x}\right)_{\tilde{x}}d\tilde{x}\Bigg\}.\end{aligned} \tag{38}$$

The combination W_s is the wave-operator from classical (i.e. isentropic) acoustic theory; W_T is also the classical wave-operator except that the isentropic speed of sound $a_0 = (\gamma RT)^{\frac{1}{2}}$ is replaced by the isothermal speed of sound $a_0\gamma^{-\frac{1}{2}} = (RT)^{\frac{1}{2}}$. The dimensionless combination $(\gamma-1)\sigma T_0^3/\gamma R\rho_0 a_0$ that appears on the right-hand side of equation (38) is the form, appropriate to a perfect gas, of one of the two dimensionless parameters that govern the combination of fluid convection and radiative transfer. Its inverse has been referred to in the Russian literature (see, for example, Adrianov & Shorin 1958) as the 'Boltzmann number', given in general by

$$N_{Bo} \equiv c_p\rho V/\sigma T^3, \tag{39a}$$

where c_p is the specific heat at constant pressure and V is a characteristic speed (equal respectively to $\gamma R/(\gamma-1)$ and a in the present instance). The combination $\alpha_0 a_0$ that also appears on the right-hand side of equation (38) has the dimensions of $(\text{time})^{-1}$ and is eventually to be compared with a characteristic time of motion of the wall. The resulting second dimensionless parameter has been called by the Russians the 'Bueger number', given generally by

$$N_{Bu} \equiv \alpha L, \tag{39b}$$

where L is a characteristic length (equal to a_0 times a characteristic time in the present application). The same combination N_{Bu}/N_{Bo} that thus, in effect, appears in equation (38) has been shown by Goulard (1959*a*) to govern stagnation-point energy-transfer in very-high-speed re-entry problems.

The structure of equation (38) is suggestive of the results that will be obtained later. If we set $1/N_{Bo} = 0$ (completely cold gas), the right-hand side of the equation disappears and a solution can then be obtained by taking $W_s = 0$. Since the functions $E_2(z)$ and $E_3(z)$ remain finite for zero values of their argument, the same situation also occurs when $\alpha_0 = 0$ (completely transparent gas). In these special cases, therefore, a solution is given by classical equilibrium acoustic theory. This is as it should be since, when the gas is either completely cold or completely transparent, no non-equilibrium radiation effects can occur. When $1/N_{Bo}$ is finite and α_0 approaches infinity (completely opaque gas), it can be seen that the right-hand side of equation (38) again goes to zero. This is because the functions $E_n(z)$ approach zero exponentially for large values of their argument and thus outweigh the effect of the α_0 that appears as a multiplying factor on the right. Thus again, a solution is given by classical acoustic theory. This is so because, although there may be intense radiation, it is immediately reabsorbed at its origin by the completely opaque gas, and no radiative transfer takes place. In the remaining limit $1/N_{Bo} \to \infty$ (infinite rate of heat transfer due to radiation), the left-hand side of equation (38) disappears and a solution can be obtained by taking either (*a*) $W_T = 0$ with $T'_w = 0$, or (*b*) $\partial W_T/\partial x = 0$ with $dT'_w/dt = (W_T)_{x=0}$. Case (*a*) corresponds to ordinary acoustic waves, except that the speed of propagation is now the isothermal instead of the classical isentropic speed of sound. This comes about because the condition of infinite heat transfer due to radiation now prevents any temperature differences from occurring in the gas. Case (*b*), in view of the relation (36), corresponds to taking T' as a function of t alone, with $T'_w(t) = T'_{x=0}(t)$, which describes a spatially uniform system with gas and wall temperature varying only with time. This is also a reasonable possibility when the rate of heat transfer due to radiation is infinite. All of the foregoing instances will appear later as limiting forms of the solution of equation (38). Which of cases (*a*) and (*b*) above will prevail when $1/N_{Bo} \to \infty$ will be seen to depend on the boundary conditions.

For small departures from the limit $\alpha_0 \to \infty$, equation (38) can be replaced by a pure differential equation by means of the Rosseland approximation (see Rosseland 1936; also Tellep & Edwards 1960). Rosseland reasoned that when reabsorption of radiation takes place in a distance over which there is little temperature change, it should be possible to replace the integrated effect of radiation by an equivalent differential term of the heat-conduction type. In the present linearized approximation this amounts to replacing Q' in equation (28) by $Q' = k_{r_0}\,\partial^2 T'/\partial x^2$, where k_{r_0} is an equivalent thermal conductivity due to radiation. Using the value $k_{r_0} = 16\sigma T_0^3/3\alpha_0$ (Rosseland 1936, p. 109) and combining equations as before, one then obtains in place of equation (38),

$$\frac{\partial W_s}{\partial t} = \frac{2}{3}\frac{8\gamma}{N_{Bo}}\frac{a_0}{\alpha_0}\frac{\partial^2 W_T}{\partial x^2}, \tag{40}$$

where
$$1/N_{Bo} = (\gamma - 1)\,\sigma T_0^3/\gamma R \rho_0 a_0. \tag{41}$$

This same equation can also be obtained from the integro-differential equation (38) by a formal limiting process as $\alpha_0 \to \infty$. The solution of equation (40) would therefore give the appropriate asymptotic form of the solution of the full equation (38).

Approximate solution of acoustic equations

For simplicity, we shall study here the solution of equation (38) only for the case of a black wall ($r_d = r_s = 0$, $\epsilon = 1$). In this case the equation reduces to

$$\frac{\partial W_s}{\partial t} = \frac{8\gamma}{N_{Bo}} \alpha_0 a_0 \left\{ -\left[R\frac{dT'_w}{dt} + (W_T)_{x=0} \right] E_2(\alpha_0 x) - \int_0^x E_2[\alpha_0(x-\tilde{x})] \left(\frac{\partial W_T}{\partial x}\right)_{\tilde{x}} d\tilde{x} + \int_x^\infty E_2[\alpha_0(\tilde{x}-x)] \left(\frac{\partial W_T}{\partial x}\right)_{\tilde{x}} d\tilde{x} \right\}. \quad (42)$$

The boundary conditions imposed by the motion and temperature variation of the wall, when transferred to $x = 0$, are respectively

$$\frac{\partial \phi}{\partial x}(0,t) = u'(0,t) = \text{a given function of } t \quad (43)$$

and

$$T'_w(t) = \text{a given function of } t. \quad (44)$$

The requirement that the disturbances must remain finite at infinity gives the further condition that

$$\phi(\infty, t) = \text{a finite quantity}. \quad (45)$$

In the present paper attention will be confined to the case of sinusoidal wall motion and temperature variation at equal frequency but with arbitrary relative phase. (Solutions for arbitrary wall motion and temperature variation can then, of course, be found by superposition.) Using complex notation we therefore take the solution for ϕ in the form

$$\phi = (RT_0/\omega)\,\mathrm{Re}\,[H(\xi)\,e^{i\omega t}], \quad (46)$$

where ω is the radian frequency of oscillation. The factor RT_0/ω is included to make the complex quantity H dimensionless. The argument of H has also been made dimensionless through division of x by the characteristic length a_0/ω to obtain the new variable

$$\xi = \omega x/a_0. \quad (47)$$

These steps lead to an equation for $H(\xi)$ containing the minimum number of parameters.

Consistent with equation (46), the boundary conditions are written

$$\frac{\partial \phi}{\partial x}(0,t) = \frac{RT_0}{a_0}\mathrm{Re}\,[H'(0)\,e^{i\omega t}] = \frac{RT_0}{a_0}\mathrm{Re}\,[A\,e^{i\omega t}], \quad (48)$$

$$\frac{dT'_w(t)}{dt} = \omega T_0\,\mathrm{Re}\,[B\,e^{i\omega t}], \quad (49)$$

and

$$\phi(\infty,t) = (RT_0/\omega)\,\mathrm{Re}\,[H(\infty)\,e^{i\omega t}] = \text{a finite quantity}, \quad (50)$$

where A and B are dimensionless complex constants, assumed to be specified. The fact that A and B are complex allows for the possibility of arbitrary phase between the motion and temperature variation of the wall. The magnitudes of A and B must be small compared with unity in conformity with the requirement of small disturbances imposed in the linearization.

Substitution of equations (46) and (49) into equation (42) and cancellation of common factors leads to the following equation for $H(\xi)$

$$H(\xi)+H''(\xi) = -i(8\gamma/N_{Bo})\,N_{Bu}\Big\{[B-H(0)-\gamma^{-1}H''(0)]\,E_2(N_{Bu}\xi) - \int_0^{\xi} E_2[N_{Bu}(\xi-\tilde{\xi})]\,[H'(\tilde{\xi})+\gamma^{-1}H'''(\tilde{\xi})]\,d\tilde{\xi} + \int_{\xi}^{\infty} E_2[N_{Bu}(\tilde{\xi}-\xi)]\,[H'(\tilde{\xi})+\gamma^{-1}H'''(\tilde{\xi})]\,d\tilde{\xi}\Big\}, \quad (51)$$

where N_{Bu} is the Bueger number previously referred to and defined here by the relation

$$N_{Bu} = \alpha_0 a_0/\omega. \quad (52)$$

As follows from equations (48) and (50), the boundary conditions for $H(\xi)$ are

$$H'(0) = A, \quad (53)$$

and

$$H(\infty) = \text{a finite quantity}. \quad (54)$$

Equation (51) is a linear integro-differential equation similar to that appearing in the Milne problem of isotropic scattering of radiation or slow neutrons (see, for example, Morse & Feshbach 1953). A method previously employed for solution of the Milne problem could therefore be used here. This method, which is exact, utilizes the Fourier-transform plus the Wiener–Hopf technique for factoring the transform. The solution appears, however, in the form of an integral expression for the transform, which is inconvenient in an initial investigation. In view of this, we shall adopt instead an approximate method that leads to a solution in closed form. The basis of this method is a device previously used in astrophysical radiation problems (see, for example, Chandrasekhar 1950), in which the continuous directional dependence of the radiation intensity is approximated by a discrete directional dependence. For the present problem, this corresponds to replacing the integral over μ in the definition

$$E_2(z) = \int_0^1 e^{-z/\mu}\,d\mu$$

(see equation (14b)) by a sum over several discrete values of μ. The approximation can be extended to any desired degree of accuracy and becomes exact in the limit of an infinite number of values of μ.

If the attenuation factor $E_2(z)$ is approximated by a sum of exponentials in the foregoing manner, the resulting integral equation can readily be solved. The solution, which appears as a sum of exponentials of complex arguments, becomes lengthy, however, as the number of terms in the approximation is increased. Here we shall adopt the simplest possible approximation, replacing the exact expression by a single exponential of the form

$$E_2(z) \simeq m\,e^{-nz}. \quad (55a)$$

The best choice for the constants m and n is open to debate. To provide the correct first-order behaviour of the solution for small departures from the limit $N_{Bu} \to \infty$, it may be desirable to use an approximation that reproduces the Rosseland form of the governing equation in the limit. It can be shown that this will be accomplished if the value of the integral

$$\int_0^1 zE_2(z)\,dz$$

is matched for the exact and approximate expressions for $E_2(z)$. This condition is found to be satisfied by the foregoing approximation if

$$E_2(z) \simeq \tfrac{1}{3}n^2 e^{-nz}, \tag{55b}$$

where n is still arbitrary. To prescribe n it might now be appropriate to impose a requirement with regard to the behaviour for small departures from the limit $N_{Bu} \to 0$. Unfortunately, however, the correct asymptotic form of the theory for this limit is not known. We shall therefore resort to the intuitive idea that the best accuracy for intermediate and small values of N_{Bu} will be attained if the exponential expression approximates $E_2(z)$ as closely as possible over its entire range. This is accomplished by means of a least-squares fit, which can be carried out analytically and leads to the requirement $n^2+n-4 = 0$. Solution of this equation gives $n = 1{\cdot}562$. The exponential approximation is therefore taken as

$$E_2(z) \simeq 0{\cdot}813\, e^{-1{\cdot}562z}. \tag{55c}$$

This approximation is compared with the exact relation in figure 2.

It will be noted from figure 2 that the exponential approximation (55) does not reproduce the infinite derivative of $E_2(z)$ at $z = 0$. For this reason the solution of the resulting approximate equation cannot be expected to lead to the singular behaviour previously anticipated in the derivatives of the physical quantities at the wall (cf. the paragraph following equation (33)). It is reasonable to suppose, however, that the values of the physical quantities themselves will be adequately approximated, even if their derivatives are not. Further discussion of the exponential approximation and of the values of m and n will be found in the concluding section.

With $E_2(z)$ approximated according to equation (55c), equation (51) becomes

$$H(\xi)+H''(\xi) = -\tfrac{1}{2}iK\beta\Big\{[B-H(0)-\gamma^{-1}H''(0)]\,e^{-\beta\xi} - \int_0^{\xi} \exp\{-\beta(\xi-\tilde{\xi})\}\,[H'(\tilde{\xi})+\gamma^{-1}H'''(\tilde{\xi})]\,d\tilde{\xi} + \int_{\xi}^{\infty} \exp\{-\beta(\tilde{\xi}-\xi)\}\,[H'(\tilde{\xi})+\gamma^{-1}H'''(\tilde{\xi})]\,d\tilde{\xi}\Big\}, \tag{56}$$

where we have for convenience introduced the notation

$$\beta = 1{\cdot}562N_{Bu}, \quad K = 8{\cdot}33\gamma/N_{Bo}. \tag{57a,b}$$

By virtue of the exponential approximation, this equation can now be converted into an ordinary differential equation by differentiating twice and then using the undifferentiated equation to remove the integral terms. The general solution of the resulting fourth-order *differential* equation is then of the form

$$H(\xi) = C_1 e^{c_1\xi} + C_2 e^{c_2\xi} + C_3 e^{c_3\xi} + C_4 e^{c_4\xi}, \quad (58)$$

where the C_j's and c_j's are complex quantities. This expression must also contain the general solution of the *integro-differential* equation (56); but, since differentiation may introduce additional solutions, it may contain some spurious

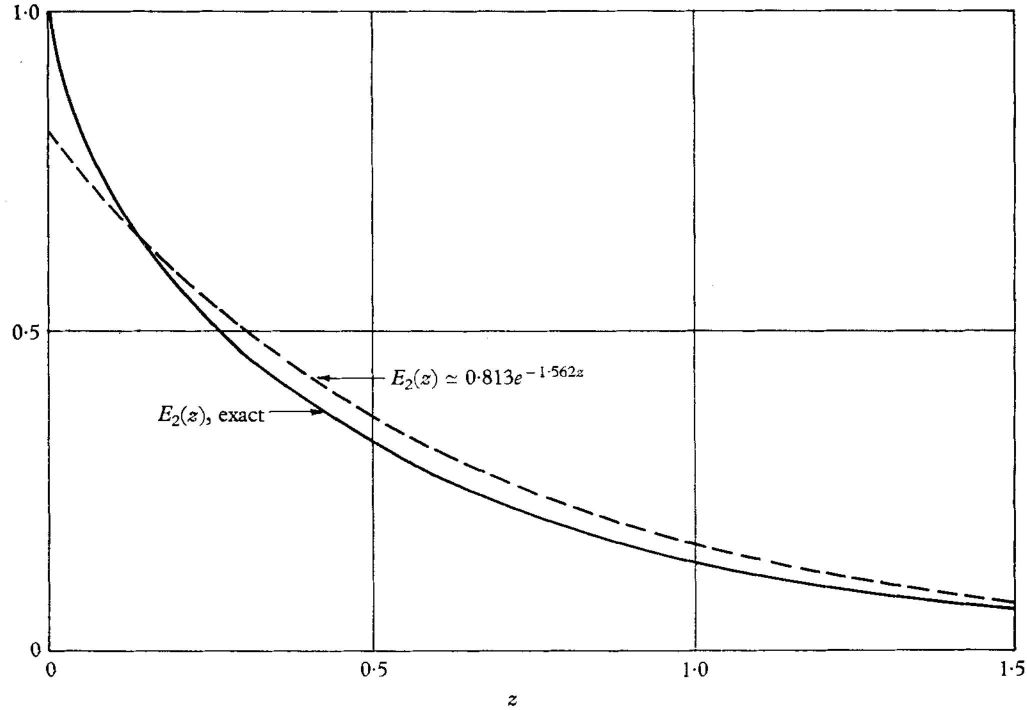

FIGURE 2. Comparison of exact and approximate values of attenuation factor.

terms. It can be verified that this is not the case here—and equations for the evaluation of the C_j's and c_j's can be obtained at the same time—by substituting equation (58) directly into equation (56). This leads, after evaluation of the resulting integrals, to the equation

$$\sum_j \left\{(1+c_j^2) - iK\beta\gamma^{-1}(\gamma+c_j^2)\left(\frac{c_j^2}{c_j^2-\beta^2}\right)\right\} C_j e^{c_j\xi} = -\tfrac{1}{2}iK\beta\left\{B - \sum_j \gamma^{-1}(\gamma+c_j^2)\left(\frac{\beta}{\beta+c_j}\right)C_j\right\} e^{-\beta\xi}. \quad (59)$$

For this equation to be satisfied for all values of ξ, it is necessary that the coefficients of each $e^{c_j\xi}$ and of $e^{-\beta\xi}$ be zero. Imposing this condition yields the five relations

$$(1+c_j^2) - iK\beta\gamma^{-1}(\gamma+c_j^2)\,c_j^2(c_j^2-\beta^2)^{-1} = 0 \quad (j = 1, 2, 3, 4) \quad (60)$$

and

$$\sum_j \gamma^{-1}(\gamma+c_j^2)\,\beta(\beta+c_j)^{-1}C_j = B. \quad (61)$$

A sixth equation relating the C_j's and c_j's can be found by substituting equation (58) into the boundary condition (53) to obtain

$$\sum_j c_j C_j = A. \tag{62}$$

The equations (60) above are seen to be four identical fourth-order algebraic equations for the c_j's. They will each have, therefore, an identical set of four complex roots that are the constants c_1, c_2, c_3, and c_4 appearing in equation (58). Since the roots are four in number, we may conclude that the complete expression (58) is indeed the general solution of the integro-differential equation (56).

At this point, a singular property of the characteristic equation (60) should be noted. For all finite, non-zero values of K and β, the situation is as just described, i.e. the equation has four roots. When either K or β is identically zero, however, the equation reduces to a second-order equation, and there are only the two roots $c_j = \pm i$. These are, in fact, the two roots appropriate to classical, equilibrium acoustic theory. In the limit as K or β tends to zero, therefore, the problem has a kind of singular-perturbation behaviour. Despite this fact, however, we shall find that the solutions of the present non-equilibrium problem go over smoothly into those of equilibrium acoustic theory at these limits. The same situation holds when $\beta \to \infty$. When $K \to \infty$, however, equation (60) remains of fourth order and has the four roots $c_j = \pm i\sqrt{\gamma}$ and $c_j = \pm 0$. As will be seen later, these matters are related to the limiting behaviour of the basic equation (38), as previously discussed.

Returning now to the consideration of equation (60) in the general case, we note that the boundary condition (54) requires that the real part of c_j be non-positive. Since equation (60) is quadratic in c_j^2, half of its roots will in general have positive real part and must therefore be excluded. If c_3 and c_4 are taken to be the two roots with positive real part, we must then take C_3 and C_4 equal to zero. Under these conditions, equations (61) and (62) provide two simultaneous linear equations for the determination of C_1 and C_2 in terms of the constants A and B and of the values of c_1 and c_2 found from equation (60).

To find c_1 and c_2, one can solve equation (60) formally to obtain

$$c_{1,2} = -\left\{\frac{-(1-\beta^2-iK\beta) \mp [(1-\beta^2-iK\beta)^2+4\beta^2(1-iK\beta/\gamma)]^{\frac{1}{2}}}{2(1-iK\beta/\gamma)}\right\}^{\frac{1}{2}}, \tag{63}$$

where the upper sign goes with c_1 and the lower with c_2. Expressions for the real and imaginary parts of c_1 and c_2 can be obtained from this relation, but they are lengthy and involved owing to the successive square roots. Furthermore, substitution of the results into equations (61) and (62) to find C_1 and C_2 leads to relations that are cumbersome to handle. Again we shall look, therefore, for simple approximations that exhibit the essential results.

A suitable approximation is suggested by noting that, when $\gamma = 1$ in equation (60), the root c_1 is given by $c_1^2 = -1$ irrespective of the values of K and β. This suggests that it might be useful to expand c_1^2 about $\gamma = 1$ as a Taylor series in powers of $(\gamma - 1)$. If we are interested only in the first power of $(\gamma - 1)$, this can be done most readily by the obvious method of setting $c_1^2 = -1 + \zeta$, where ζ is a small quantity of first order in $(\gamma - 1)$, and substituting this expression into equation (60). Neglect of terms in ζ^2 then leads to an equation that can be solved for ζ. A considerable improvement on the Taylor expansion can be obtained, however, by noting from equations (60) or (63) that c_1^2 varies only from -1 to $-\gamma$ as K covers the full range from 0 to ∞. This suggests trying an expansion in which the first term has the average of these two values of c_1^2 in place of the simple value of -1 used before. We therefore set

$$c_1^2 = -\tfrac{1}{2}(\gamma+1)+\zeta, \tag{64}$$

where ζ is again of order $(\gamma-1)$, and proceed to substitute into equation (60) as before. This leads finally to the result

$$c_1^2 = -\frac{\gamma+1}{2}\left[1-\left(\frac{\gamma-1}{\gamma+1}\right)\frac{1+ia}{1-ia}+O(\gamma-1)^2\right], \tag{65}$$

where
$$a \equiv \frac{K\beta}{\gamma[1+2\beta^2/(\gamma+1)]}. \tag{66}$$

The corresponding expression for c_2^2 can be found by factoring $(c_j^2-c_1^2)$ out of the left-hand side of equation (60), i.e. by dividing the left-hand side by $(c_j^2-c_1^2)$ where c_1^2 is given by equation (65). This leads to the result

$$c_2^2 = \frac{\beta^2}{1-iK\beta/\gamma}\left[1+2\left(\frac{\gamma-1}{\gamma+1}\right)\frac{ia}{1-ia}+O(\gamma-1)^2\right]. \tag{67}$$

Comparison of limiting values obtained from these equations with corresponding values obtained from the exact equation (63) shows, surprisingly enough, that equations (65) and (67), with terms of $O(\gamma-1)^2$ neglected, give correctly the exact limiting values not only as γ approaches 1 but also in all four limits as K and β approach 0 and ∞. It is reasonable to suppose therefore, particularly in view of the relatively small range covered by c_1^2, that they will give a good approximation at intermediate values of K and β. Spot checks at particular values of K and β and with $\gamma = \frac{7}{5}$ show an error of less than 5 % as compared with exact values obtained from equation (63).

The values of c_1 and c_2 are now readily found by taking the square root of equations (65) and (67) to obtain

$$c_1 = -i\left(\frac{\gamma+1}{2}\right)^{\frac{1}{2}}\left[1-\frac{1}{2}\left(\frac{\gamma-1}{\gamma+1}\right)\frac{1+ia}{1-ia}+O(\gamma-1)^2\right], \tag{68}$$

$$c_2 = -\frac{\beta}{(1-iK\beta/\gamma)^{\frac{1}{2}}}\left[1+\left(\frac{\gamma-1}{\gamma+1}\right)\frac{ia}{1-ia}+O(\gamma-1)^2\right]. \tag{69}$$

These equations are no longer exact in the limit as K and β tend to 0 and ∞. For $\gamma = \frac{7}{5}$, however, the limiting values are in error by less than 0·4%, and the spot checks at intermediate values of K and β show an accuracy comparable to that cited above.

The nature of the solution that has been obtained can now be examined by returning to expression (46) for the potential ϕ. At this point it is convenient to introduce symbols for the real and imaginary parts of c_1 and c_2 according to the relations

$$c_1 = -(\delta_1 + i\lambda_1), \tag{70}$$

$$c_2 = -(\delta_2 + i\lambda_2), \tag{71}$$

where the dimensionless δ's and λ's are positive real quantities. With H replaced by means of equation (58), equation (46) for ϕ can then be written

$$\phi = (RT_0/\omega)\,\mathrm{Re}\,[C_1 \exp(-\delta_1 \omega x/a_0)\exp\{i\omega(t-\lambda_1 x/a_0)\} + C_2 \exp(-\delta_2 \omega x/a_0)\exp\{i\omega(t-\lambda_2 x/a_0)\}]. \tag{72}$$

Equation (72) represents a superposition of two damped waves travelling away from the wall. The damping of each wave increases with the corresponding value of δ. The speed of the wave is determined by the value of λ—specifically, it is equal to a_0/λ.

Expressions for the δ's and λ's can be obtained from equations (68) and (69). With the notation

$$\Gamma \equiv (\gamma-1)/(\gamma+1) \quad \text{and} \quad b \equiv K\beta/\gamma, \tag{73}$$

they are as follows:

$$\delta_1 = \{\tfrac{1}{2}(\gamma+1)\}^{\frac{1}{2}}\,\Gamma a/(1+a^2) + O(\gamma-1)^2, \tag{74a}$$

$$\lambda_1 = \{\tfrac{1}{2}(\gamma+1)\}^{\frac{1}{2}}\{1-\tfrac{1}{2}\Gamma(1-a^2)/(1+a^2)\} + O(\gamma-1)^2, \tag{74b}$$

$$\delta_2 = \frac{\beta}{\sqrt{2}}\left\{\left(\frac{1+(1+b^2)^{\frac{1}{2}}}{1+b^2}\right)^{\frac{1}{2}}\left(1-\Gamma\frac{a^2}{1+a^2}\right) - \Gamma\left(\frac{-1+(1+b^2)^{\frac{1}{2}}}{1+b^2}\right)^{\frac{1}{2}}\frac{a}{1+a^2}\right\} + O(\gamma-1)^2, \tag{75a}$$

$$\lambda_2 = \frac{\beta}{\sqrt{2}}\left\{\left(\frac{-1+(1+b^2)^{\frac{1}{2}}}{1+b^2}\right)^{\frac{1}{2}}\left(1-\Gamma\frac{a^2}{1+a^2}\right) + \Gamma\left(\frac{1+(1+b^2)^{\frac{1}{2}}}{1+b^2}\right)^{\frac{1}{2}}\frac{a}{1+a^2}\right\} + O(\gamma-1)^2. \tag{75b}$$

To examine the nature of the two waves, the values of the δ's and λ's for $\gamma = \frac{7}{5}$ have been plotted in figure 3 as functions of β for three discrete values of K, corresponding to very high, intermediate, and very low temperatures. (For convenience, the definition of the variables is repeated in the figure caption.) The unusual nature of these plots may require the following explanation. The absorption parameter β has been taken as the primary variable, and high- and low-temperature values of K short of the limits ∞ and O have been used, in order to show the boundary-layer-like behaviour of the results for certain combinations of the variables. Actual distances X on the horizontal scale have been plotted according to the formula $X = 2\beta/(1+\beta)$ for the range of values of β from $\beta = \epsilon$ to $\beta = 1/\epsilon$, where $\epsilon \ll 1$. Thus, X goes essentially from 0 to 2 as β goes from ϵ to $1/\epsilon$. Outside this range, at both extremes of β, the horizontal scale has been expanded arbitrarily to provide a representation of the boundary-layer-like behaviour near the limits. The representation here is thus qualitative in the horizontal direction but remains quantitative vertically. In the uppermost plots of figure 3, ϵ is taken to be of an order larger than $1/K$. These plots can be regarded alternatively as representing the limit $K \to \infty$ when the arbitrarily expanded regions outside ϵ and $1/\epsilon$ are squeezed down to zero width. In the lowermost plots, ϵ is taken to be of an order larger than K. These plots can be thought of as representing the limit $K \to 0$ when the expanded regions are eliminated. The fact that the vertical scale for δ_1 in the left-hand plots is different from that for the other quantities should be noted. In the right-hand plots for $K \gg 1$ and $\ll 1$, the quantities that are small in the left-hand plots have been suitably amplified to make their variation more apparent.

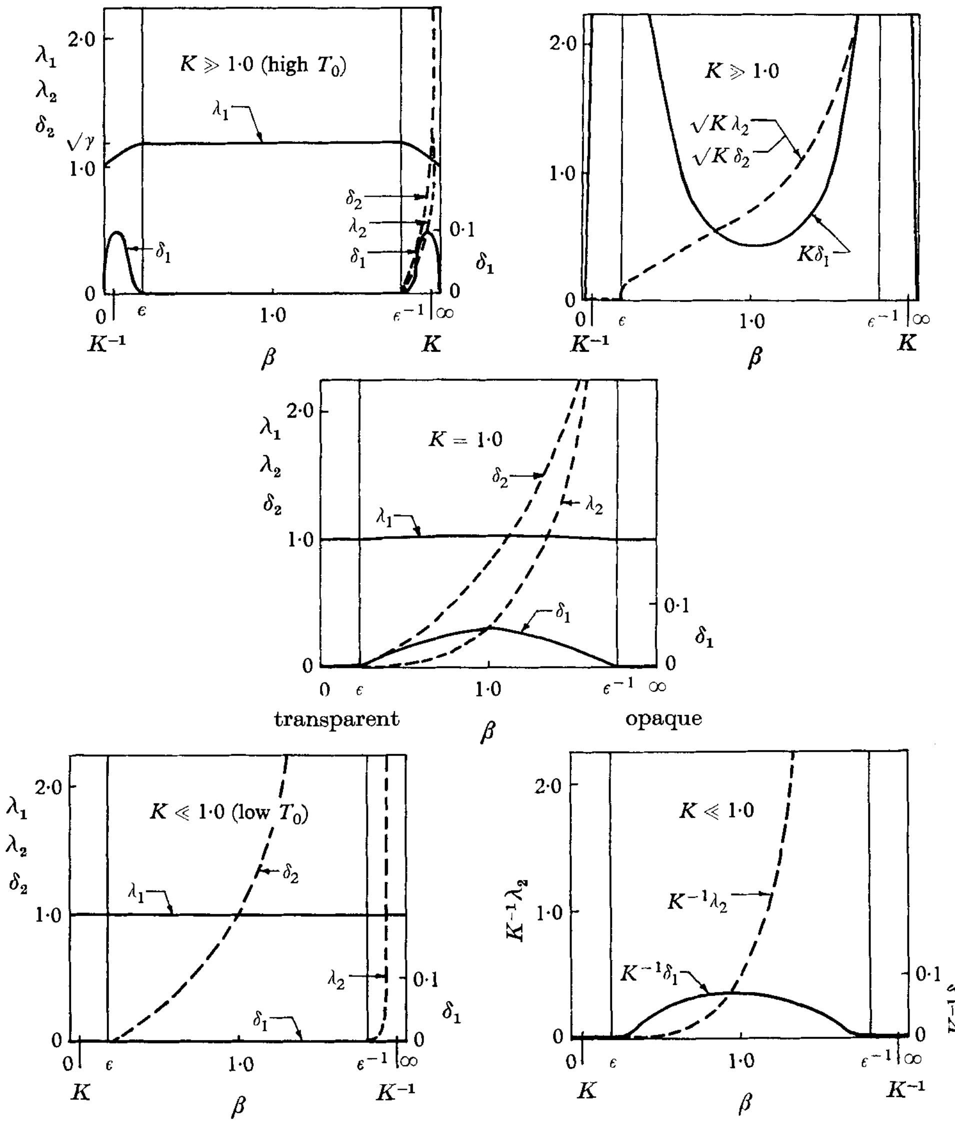

FIGURE 3. Dimensionless wave-speed and damping factors λ and δ as functions of the dimensionless absorption and temperature parameters β and K. For the meaning of λ and δ, see equation (73); the definition of independent variables is $\beta \equiv 1{\cdot}562\alpha_0 a_0/\omega$, $K = 8{\cdot}33(\gamma-1)\,\sigma T_0^3/R\rho_0 a_0$, where α_0 is the absorption coefficient, a_0 the isentropic speed of sound, T_0 the static temperature, and ρ_0 the mass density, all in the undisturbed fluid, ω is the frequency of oscillation of the position and temperature of the wall, and σ is the Stefan–Boltzmann constant, R the gas constant per unit mass, and γ the ratio of specific heats. Other things being equal, the greater the value of β, the more absorbent the gas; the greater the value of K, the higher the temperature level of the system. ——, C_1 term; ----, C_2 term.

It is apparent from figure 3 that the C_1-term in the solution (72) is essentially a classical sound-wave, but with a slightly altered speed and a small amount of damping as the result of radiation. For a fixed value of β other than 0 and ∞, the speed of this wave goes monotonically from the isentropic sound speed a_0 ($\lambda_1 = 1$) to the isothermal sound speed $a_0/\sqrt{\gamma}$ ($\lambda_1 = \sqrt{\gamma}$) as K goes from 0 to ∞. Over the same range, which corresponds to a variation from very low to very high temperatures, the damping goes from zero to small positive values and then back to zero. When β is identically zero, the gas is completely transparent, and the wave is unaffected by radiation, i.e. the speed is a_0 and the damping is zero regardless of the value of K. For $\beta = \infty$, the gas is completely opaque, and again the classical result is obtained for all values of K. The behaviour of this modified classical wave thus agrees in all of the various limits with that anticipated following equation (38), where the underlying physical reasons were explained. The limiting results can also be identified with the special roots of the characteristic equation (60) that were previously discussed—i.e. the limiting values for $K \to 0$, $\beta \to 0$, and $\beta \to \infty$ correspond to the single negative root of the equation in its singular limit; the values for $K \to \infty$ correspond to the first of the two negative roots in the non-singular limit.

The results for the modified classical wave at large values of K (very high temperatures) are somewhat curious. As shown in figure 3, the speed of the wave goes from a_0 to $a_0/\sqrt{\gamma}$ as β increases over a narrow range near $\beta = K^{-1}$. At the same time the damping reaches a maximum and then returns to zero. A similar transition of the wave speed back to a_0 occurs for large values of β near $\beta = K$, accompanied again by a sharp local peak in the damping.

It is appropriate to recall here that the first-order departure from an infinite value of β in the present results corresponds to the Rosseland theory as a result of the particular choice (55*b*) for the exponential approximation. Working directly with the Rosseland-type equation (40) from the beginning leads to results similar to those given in figure 3, except that the dependence of the damping and wave speed on β and K is distorted at all except very large values of β. Also, the eventual return with decreasing β to undamped waves with the classical speed a_0, regardless of the value of K, is not predicted.

In contrast to the Rosseland theory, the work of Stokes (1851) applies, in some imprecise way, for small values of β. Stokes realized at the outset that high levels of radiative transfer can result in an isothermal speed of sound. To see if this can occur under ordinary atmospheric conditions he analysed a model that assumes, in effect, that the radiative transfer takes place between each element of gas and a reservoir at the undisturbed temperature. This model can be justified physically for small β. Although not in possession of data on the absorption coefficient of air, Stokes reasoned essentially that β is small because air is transparent. He also reasoned, in effect, that K must be small because when a small portion of a mass of air at atmospheric conditions is heated in some fashion, it does not immediately fall back to ambient temperature by radiation. With present-day knowledge, we can verify these deductions quantitatively. For air at atmospheric temperature, we find, in fact, that $\beta \leqslant 1/30f$, where f is the frequency of the waves in cycles per second, and that $K \simeq 4 \times 10^{-5}$. For these small values of β and K, the results for the modified classical wave in figure 3 show, in agreement with the final conclusions of Stokes's analysis, that the sound speed is a_0 and the damping is negligible.

Turning now to the C_2-term in equation (72), we see that this term represents a type of sound wave that has no counterpart in classical acoustic theory. (Since equation (40) is of fourth order in derivatives of ϕ, this wave could also be obtained with the Rosseland theory; it could not, however, have appeared in Stokes's work.) Contrary to the situation with the modified classical wave, the speed and damping of this radiation-induced wave are strong functions of K and β. At all values of K, the speed varies from infinite ($\lambda_2 = 0$) at $\beta = 0$ to zero ($\lambda_2 = \infty$) at $\beta = \infty$. This is so even though the radiative signal, on which the wave depends, itself travels at infinite speed. The damping of the wave has a similarly large variation, going in all cases from zero at $\beta = 0$ to infinity at $\beta = \infty$. The precise way in which the variations with β occur, however, has a marked dependence on K. The result is that, for a fixed value of $\beta > 1$, the damping goes monotonically from a sizeable positive value at $K = 0$ to zero at $K = \infty$; the wave speed goes at the same time from an infinite value to finite values and then back again to infinity. Over most of the range of the variables, the radiation-induced wave has a greater damping and a higher speed than the modified classical wave. At sufficiently high temperatures, however, the damping of the two waves may be comparably small; and, in a sufficiently absorbent gas, the speeds may be essentially equal.

It may be noted that at the limits $K \to 0$, $\beta \to 0$ and $\beta \to \infty$, in which the C_1-term was seen to reduce to the classical acoustic wave, the results of figure 3 show specific, well-defined values for λ_2 and δ_2. It follows, for example, that, in the limit $K \to 0$ ($\lambda_2 = 0$, $\delta_2 > 0$), the radiation-induced wave has the form $C_2 \exp(-\delta_2 \omega x/a_0) \exp(i\omega t)$, which describes a damped standing wave of infinite wavelength. Such a wave, however, has no counterpart in classical acoustic theory. The existence of λ_2 and δ_2 in all of the aforementioned limits is associated, in fact, with the roots of the characteristic equation (60) that disappear discontinuously when K and β are identically equal to their limiting values. How the full solution goes over into that of classical acoustic theory will be apparent only after we have examined the role played by the boundary conditions.

The dependence of the solution on the boundary conditions can best be shown by studying the response of the gas to the motion of a constant-temperature wall and to the temperature variation of a motionless wall. To do this, c_1 and c_2 from equations (68) and (69) are substituted into equations (61) and (62) and these equations then solved for C_1 and C_2 for the special cases $B = 0$ and $A = 0$. When expressed in terms of amplitude ratios, and with the notation

$$r_1 = (1+b^2)^{\frac{1}{2}}, \quad r_2 = \{(1+\beta^2/\gamma)^2+b^2\}^{\frac{1}{2}}, \quad r_3 = \{[1-r_1^{\frac{1}{2}}\cos(\tfrac{1}{2}\theta_1)]^2 + r_1\sin^2(\tfrac{1}{2}\theta_1)\}^{\frac{1}{2}},$$
$$r_4 = \{\beta^2+\tfrac{1}{2}(\gamma+1)\}^{\frac{1}{2}}, \quad r_5 = (1+a^2)^{\frac{1}{2}}, \quad \theta_1 = -\tan^{-1} b,$$

the results are as follows: for pure wall motion,

$$\left(\frac{|C_1|}{|A|}\right)_{B=0} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}}\left(1+\tfrac{1}{2}\Gamma\frac{1-a^2}{1+a^2}\right)+O(\gamma-1) \tag{76a}$$

and

$$\left(\frac{|C_2|}{|A|}\right)_{B=0} = \frac{\gamma-1}{\gamma}\left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}}\frac{\beta r_1^{\frac{1}{4}} r_3}{r_2 r_4 r_5}+O(\gamma-1)^2; \tag{76b}$$

for pure wall-temperature variation,

$$\left(\frac{|C_1|}{|B|}\right)_{A=0} = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}} \frac{\beta r_3}{r_2}\left(1+\tfrac{1}{2}\Gamma\frac{1-3a^2}{1+a^2}\right)+O(\gamma-1) \tag{77a}$$

and

$$\left(\frac{|C_2|}{|B|}\right)_{A=0} = \frac{r_1^{\frac{1}{2}} r_3}{r_2}+O(\gamma-1). \tag{77b}$$

(The corresponding phase relationships could also be obtained, but they give nothing of particular interest.) If $\phi_1(x,t)$ is the part of the solution (72) corresponding to the C_1-term, the ratios for C_1 can be written

$$\frac{|C_1|}{|A|} = \frac{\omega}{a_0}\frac{[\phi_1(0,t)]_{\max}}{(dx_w/dt)_{\max}}, \quad \frac{|C_1|}{|B|} = \frac{\omega^2}{R}\frac{[\phi_1(0,t)]_{\max}}{(dT'_w/dt)_{\max}},$$

and similarly for C_2. The ratios (76) and (77) thus provide a dimensionless measure of the intensity of response of a given wave at the wall compared with the intensity of the wall motion or temperature variation. It will be observed that the error in equations (76*a*), (77*a*), and (77*b*) is of $O(\gamma-1)$ instead of $O(\gamma-1)^2$ as in other relations. This is due to the neglect here of a small but very complicated term of $O(\gamma-1)$; this term is identically zero at the limiting values of K and β, and spot checks show it to be less than 0·02 elsewhere.

Results obtained from equations (76) and (77) are given in figures 4 and 5 on the same basis as before. These results show that, whenever radiative non-equilibrium is present, either type of wall disturbance gives rise to both kinds of waves. For pure wall motion (figure 4), the C_1-wave predominates at all conditions, but the C_2-wave is present to some extent. For pure wall-temperature variation (figure 5), the C_2-wave predominates at very high temperatures, but at lower temperatures the two waves are of comparable intensity. One must therefore resist the temptation to make a categorical statement that the C_1-waves are due to wall motion and the C_2-waves to temperature variation. Actually, the two waves are coupled through the thermodynamics of the gas. Any disturbance, even though it may tend primarily to excite one type of wave, must also give rise to the other.

The behaviour of the complete solution in the various limits is now clear from the results of figures 4 and 5. In the three limits $K \to 0$, $\beta \to 0$, and $\beta \to \infty$, the following results hold:

$$\text{for } B=0, \quad |C_1| \to |A|, \quad |C_2| \to 0; \qquad \text{for } A=0, \quad |C_1| \to 0, \quad |C_2| \to 0.$$

We see that the C_2-wave disappears, irrespective of the limiting behaviour of λ_2 and δ_2 as previously noted. We thus obtain, in these limits, classical acoustic theory, in which wall motion produces only a C_1-wave and wall-temperature variation has no effect whatsoever. The implications of the basic integro-differential equation (38) in these limits are thus realized. In the limit $K \to \infty$, the following situation prevails:

$$\text{for } B=0, \quad |C_1| \to |A|/\sqrt{\gamma}, \quad |C_2| \to 0; \qquad \text{for } A=0, \quad |C_1| \to 0, \quad |C_2| \to |B|.$$

Here, as in the previous limits, pure wall motion produces only a C_1-wave, though, as we have seen in figure 3, the wave speed is now the isothermal speed of sound. This corresponds to case (*a*) in the discussion of this limit following equation (38). In contrast to the previous limits, however, pure wall-temperature variation now also has an effect, producing a pure C_2-term. Since

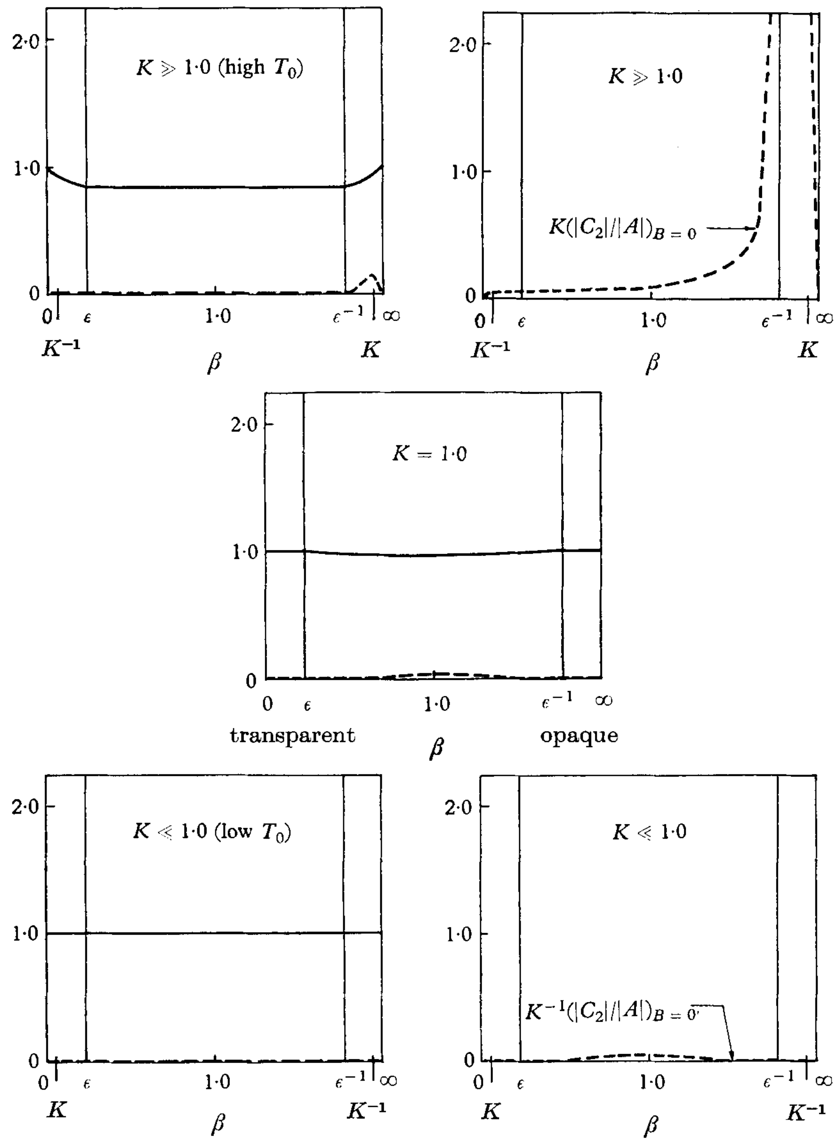

FIGURE 4. Dimensionless measure of intensity of a given wave at the wall as compared with intensity of wall motion when wall temperature is constant. For definition and interpretation of independent variables β and K, see figure 3. ——, $(|C_1|/|A|)_{B=0}$; ----, $(|C_2|/|A|)_{B=0}$.

$\lambda_2 = \delta_2 = 0$ as $K \to \infty$ (see figure 3), this term here represents a field that is uniform in x and varies only with t. This corresponds to case (*b*) in the discussion following equation (38). Thus, in the limit of infinitely high temperatures, the infinite heat transfer results in an uncoupling of the effects of wall motion and temperature variation. (It is interesting to note that this limit at which both types of waves can exist is also the limit at which the characteristic equation (60) is non-singular.)

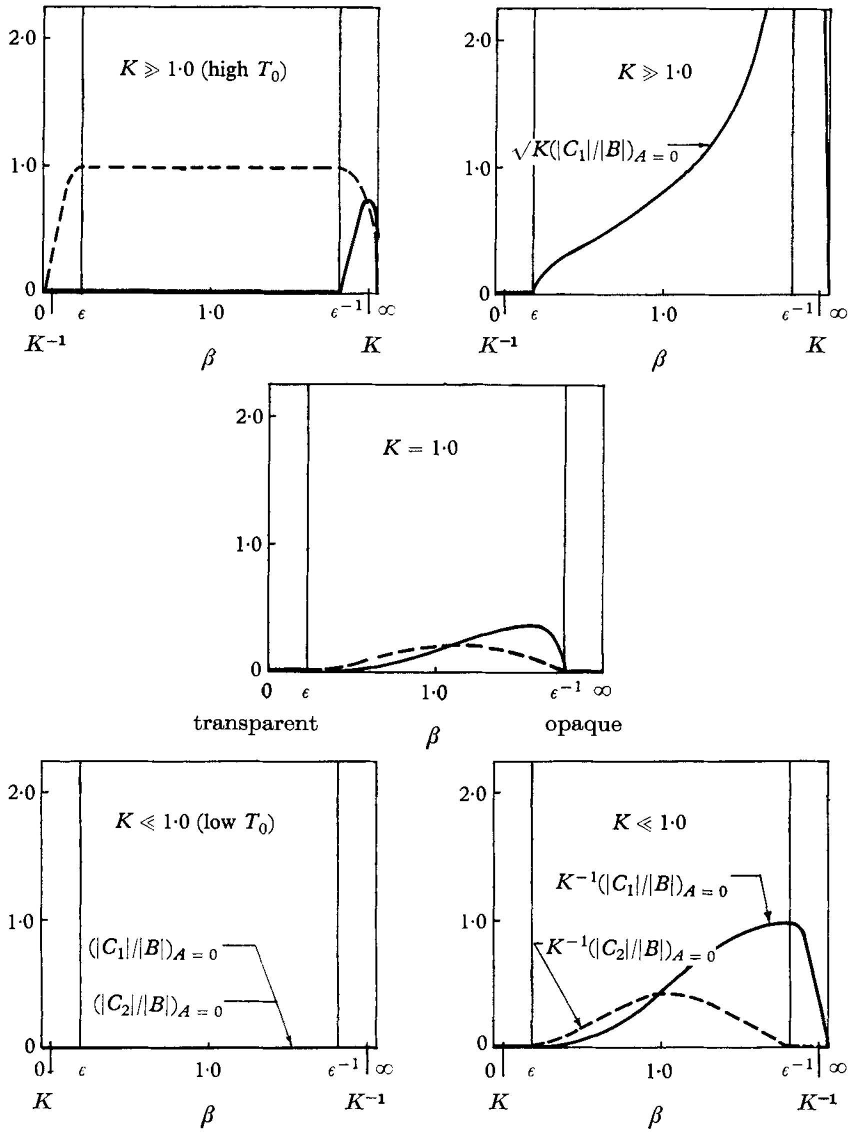

FIGURE 5. Dimensionless measure of intensity of a given wave at the wall as compared with intensity of wall-temperature variation when wall is motionless. For definition and interpretation of independent variables β and K, see caption to figure 3. ——, $(|C_1|/|B|)_{A=0}$; - - - -, $(|C_2|/|B|)_{A=0}$.

Concluding remarks

The foregoing work rests, of course, on the use of the exponential approximation (55) for the attenuation factor E_2. It is apparent, however, that the precise choice of the constants m and n in the approximation is a secondary matter. As has been seen, the exponential approximation yields, at the various limits, the results anticipated on the basis of the *complete* integro-differential equation (38)—and this would be true irrespective of the values of m and n. The only effect of changing these values would be to change the constants in the equations (57) that relate the numbers N_{Bu} and N_{Bo} to the parameters β and K; the subsequent analysis in terms of these latter parameters would be unaltered. The qualitative results of the analysis would thus remain unchanged, and only the quantitative findings in terms of the original physical variables would be affected.

A more serious question is the restriction to only one exponential term in the approximation. The use of more than one term of this kind leads to a slight alteration in the dependence of the speed and damping of the present C_1- and C_2-waves on the temperature and absorption of the gas. This comes about through a modification of the functional relationship of N_{Bu} and N_{Bo} to β and K. In addition, more waves of the C_2-type are introduced, all with an almost infinite speed but with varying damping. In the limit of an infinite number of exponential terms, where an exact solution is obtained, there are an infinite number of such waves. It seems reasonable to suppose, however, that the contribution of the additional component waves needed to correct the present result will generally be small compared with that of the waves studied here. The present solution should therefore represent the essential character of the phenomena.

Proof of this assertion for large distances from the wall can be obtained, in fact, by studying equation (51) for large values of ξ. It is found that, for sufficiently high values of N_{Bu}, an asymptotic solution for $H(\xi)$ can be found in exponential form even with the exact integral representation (14*b*) for E_2. The resulting characteristic equation for the c_j differs somewhat from equation (60), but the final results agree qualitatively with those of the present solution based on the exponential approximation. For small values of N_{Bu} the situation is more complicated, but again it can be shown that the present approximate solution agrees essentially with the exact asymptotic behaviour.

From the physical point of view, the most critical approximation in the present work is that of the grey gas, i.e. that the absorption coefficient α_ν is independent of ν. Actually, this unrealistic assumption is not essential, and the present results for the case of a black wall can be obtained on a less restrictive basis. For this case, an integro-differential equation formally identical to equation (42) can still be derived, even without the grey-gas assumption. The only difference is that the attenuation factors E_2 must be replaced by a function of x that is given as a complicated double integral over ν and μ with $\alpha_{\nu 0}$ appearing in the integrand ($\alpha_{\nu 0}$ being the value of α_ν in the undisturbed gas). This function, rather than E_2, is then approximated by a single exponential term, with the constants m and n determined on the same basis as before. The approximation procedure of the present paper, in fact, is reproduced as a special case when $\alpha_{\nu 0}$ is made a constant. The final outcome of the generalized approach is that equation (56) for $H(\xi)$ and all of the results of the analysis in terms of β and K are obtained precisely as before. The only change is that the numerical coefficients in equations (57) relating β and K to N_{Bu} and N_{Bo} now depend on the variation of $\alpha_{\nu 0}$ with ν. The details of these matters will be given elsewhere. An approach of this kind would be desirable in any attempt to check the present results against experiment.

Before concluding, the relationship of the present work to that of Prokofyev, mentioned in the introduction, should be noted. Prokofyev considers a gas of infinite, rather than semi-infinite extent, so that his results correspond in a sense to the situation for large ξ discussed in the third paragraph of these remarks. In fact, when effects due to viscosity, thermal conductivity, etc., are discarded and the grey-gas approximation introduced, Prokofyev's characteristic equation (Prokofyev 1960) agrees identically with the characteristic equation that arises in the asymptotic solution found on the basis of the exact expression for E_2. The analytical results that Prokofyev gives for large and small values of the various parameters, when also specialized to the present gas model, agree essentially with those obtained here for the modified classical wave. Although the radiation-induced wave is also implicit in Prokofyev's characteristic equation, he takes no notice of this fact.

Finally, results analogous to those presented here have also been found for acoustic waves in an infinite expanse of a viscous, heat-conducting gas (Truesdell 1953) and for dilational waves in an infinite thermoelastic solid (Deresiewicz 1957). In both of these cases, the exact governing equations are purely differential equations that can be shown (provided viscosity is ignored in Truesdell's case) to be formally identical to the approximate Rosseland-type equation (40) of the present paper. Both analyses again reveal the existence of two waves, one a modification of the classical equilibrium wave and the other a wave induced by the dissipative effects.

The authors are indebted to Prof. Milton D. Van Dyke for valuable criticism and discussion and to Dr Chi-chang Chao for pointing out the analogous results from thermoelasticity. The work was done as part of a research programme being conducted in the Department of Aeronautical Engineering, Stanford University under a grant from the National Science Foundation.

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